# ON CONOIDS AND SPHEROIDS.

## Introduction*.

"ARCHIMEDES to Dositheus greeting.

In this book I have set forth and send you the proofs of the remaining theorems not included in what I sent you before, and also of some others discovered later which, though I had often tried to investigate them previously, I had failed to arrive at because I found their discovery attended with some difficulty. And this is why even the propositions themselves were not published with the rest. But afterwards, when I had studied them with greater care, I discovered what I had failed in before.

Now the remainder of the earlier theorems were propositions concerning the right-angled conoid [paraboloid of revolution]; but the discoveries which I have now added relate to an obtuse-angled conoid [hyperboloid of revolution] and to spheroidal figures, some of which I call *oblong* (παραμάκεα) and others *flat* (ἐπιπλατέα).

I. Concerning the *right-angled conoid* it was laid down that, if a section of a right-angled cone [a parabola] be made to revolve about the diameter [axis] which remains fixed and

* The whole of this introductory matter, including the definitions, is translated literally from the Greek text in order that the terminology of Archimedes may be faithfully represented. When this has once been set out, nothing will be lost by returning to modern phraseology and notation. These will accordingly be employed, as usual, when we come to the actual propositions of the treatise.

return to the position from which it started, the figure comprehended by the section of the right-angled cone is called a **right-angled conoid,** and the diameter which has remained fixed is called its **axis,** while its **vertex** is the point in which the axis meets (ἅπτεται) the surface of the conoid. And if a plane touch the right-angled conoid, and another plane drawn parallel to the tangent plane cut off a segment of the conoid, the **base** of the segment cut off is defined as the portion intercepted by the section of the conoid on the cutting plane, the **vertex** [of the segment] as the point in which the first plane touches the conoid, and the **axis** [of the segment] as the portion cut off within the segment from the line drawn through the vertex of the segment parallel to the axis of the conoid.

The questions propounded for consideration were

(1) why, if a segment of the right-angled conoid be cut off by a plane at right angles to the axis, will the segment so cut off be half as large again as the cone which has the same base as the segment and the same axis, and

(2) why, if two segments be cut off from the right-angled conoid by planes drawn in any manner, will the segments so cut off have to one another the duplicate ratio of their axes.

II. Respecting the *obtuse-angled conoid* we lay down the following premisses. If there be in a plane a section of an obtuse-angled cone [a hyperbola], its diameter [axis], and the nearest lines to the section of the obtuse-angled cone [*i.e.* the asymptotes of the hyperbola], and if, the diameter [axis] remaining fixed, the plane containing the aforesaid lines be made to revolve about it and return to the position from which it started, the nearest lines to the section of the obtuse-angled cone [the asymptotes] will clearly comprehend an isosceles cone whose vertex will be the point of concourse of the nearest lines and whose axis will be the diameter [axis] which has remained fixed. The figure comprehended by the section of the obtuse-angled cone is called an **obtuse-angled conoid** [hyperboloid of revolution], its **axis** is the diameter which has remained fixed, and its **vertex** the point in which the axis meets the surface of the conoid. The cone comprehended by the nearest lines to the section of the obtuse-angled cone is called [the cone] **enveloping the conoid** (περιέχων τὸ κωνοειδές), and the straight line between the vertex of the conoid and the vertex of the cone enveloping the conoid is called [the line] **adjacent to the axis** (ποτεοῦσα τῷ ἄξονι). And if a plane touch the obtuse-angled conoid, and another plane drawn parallel to the tangent plane cut off a segment of the conoid, the **base** of the segment so cut off is defined as the portion intercepted by the section of the conoid on the cutting plane, the **vertex** [of the segment] as the point of contact of the plane which touches the conoid, the **axis** [of the segment] as the portion cut off within the segment from the line drawn through the vertex of the segment and the vertex of the cone enveloping the conoid; and the straight line between the said vertices is called **adjacent to the axis.**

Right-angled conoids are all similar; but of obtuse-angled conoids let those be called similar in which the cones enveloping the conoids are similar.

The following questions are propounded for consideration,

(1) why, if a segment be cut off from the obtuse-angled conoid by a plane at right angles to the axis, the segment so cut off has to the cone which has the same base as the segment and the same axis the ratio which the line equal to the sum of the axis of the segment and three times the line adjacent to the axis bears to the line equal to the sum of the axis of the segment and twice the line adjacent to the axis, and

(2) why, if a segment of the obtuse-angled conoid be cut off by a plane not at right angles to the axis, the segment so cut off will bear to the figure which has the same base as the segment and the same axis, being a segment of a cone* (ἀπότμαμα κώνου), the ratio which the line equal to the sum of the axis of the segment and three times the line adjacent to the axis bears to the line equal to the sum of the axis of the segment and twice the line adjacent to the axis.

* A *segment of a cone* is defined later (p. 104).

III. Concerning spheroidal figures we lay down the following premisses. If a section of an acute-angled cone [ellipse] be made to revolve about the greater diameter [major axis] which remains fixed and return to the position from which it started, the figure comprehended by the section of the acute-angled cone is called an **oblong spheroid** (παραμᾶκες σφαιροειδές). But if the section of the acute-angled cone revolve about the lesser diameter [minor axis] which remains fixed and return to the position from which it started, the figure comprehended by the section of the acute-angled cone is called a **flat spheroid** (ἐπιπλατὺ σφαιροειδές). In either of the spheroids the **axis** is defined as the diameter [axis] which has remained fixed, the **vertex** as the point in which the axis meets the surface of the spheroid, the **centre** as the middle point of the axis, and the **diameter** as the line drawn through the centre at right angles to the axis. And, if parallel planes touch, without cutting, either of the spheroidal figures, and if another plane be drawn parallel to the tangent planes and cutting the spheroid, the **base** of the resulting segments is defined as the portion intercepted by the section of the spheroid on the cutting plane, their **vertices** as the points in which the parallel planes touch the spheroid, and their **axes** as the portions cut off within the segments from the straight line joining their vertices. And that the planes touching the spheroid meet its surface at one point only, and that the straight line joining the points of contact passes through the centre of the spheroid, we shall prove. Those spheroidal figures are called **similar** in which the axes have the same ratio to the 'diameters.' And let segments of spheroidal figures and conoids be called **similar** if they are cut off from similar figures and have their bases similar, while their axes, being either at right angles to the planes of the bases or making equal angles with the corresponding diameters [axes] of the bases, have the same ratio to one another as the corresponding diameters [axes] of the bases.

The following questions about spheroids are propounded for consideration,

(1) why, if one of the spheroidal figures be cut by a plane

through the centre at right angles to the axis, each of the resulting segments will be double of the cone having the same base as the segment and the same axis; while, if the plane of section be at right angles to the axis without passing through the centre, (*a*) the greater of the resulting segments will bear to the cone which has the same base as the segment and the same axis the ratio which the line equal to the sum of half the straight line which is the axis of the spheroid and the axis of the lesser segment bears to the axis of the lesser segment, and (*b*) the lesser segment bears to the cone which has the same base as the segment and the same axis the ratio which the line equal to the sum of half the straight line which is the axis of the spheroid and the axis of the greater segment bears to the axis of the greater segment;

(2) why, if one of the spheroids be cut by a plane passing through the centre but not at right angles to the axis, each of the resulting segments will be double of the figure having the same base as the segment and the same axis and consisting of a segment of a cone*.

(3) But, if the plane cutting the spheroid be neither through the centre nor at right angles to the axis, (*a*) the greater of the resulting segments will have to the figure which has the same base as the segment and the same axis the ratio which the line equal to the sum of half the line joining the vertices of the segments and the axis of the lesser segment bears to the axis of the lesser segment, and (*b*) the lesser segment will have to the figure with the same base as the segment and the same axis the ratio which the line equal to the sum of half the line joining the vertices of the segments and the axis of the greater segment bears to the axis of the greater segment. And the figure referred to is in these cases also a segment of a cone*.

When the aforesaid theorems are proved, there are discovered by means of them many theorems and problems.

Such, for example, are the theorems

(1) that similar spheroids and similar segments both of

* See the definition of a *segment of a cone* (ἀπότμαμα κώνου) on p. 104.

spheroidal figures and conoids have to one another the triplicate ratio of their axes, and

(2) that in equal spheroidal figures the squares on the 'diameters' are reciprocally proportional to the axes, and, if in spheroidal figures the squares on the 'diameters' are reciprocally proportional to the axes, the spheroids are equal.

Such also is the problem, From a given spheroidal figure or conoid to cut off a segment by a plane drawn parallel to a given plane so that the segment cut off is equal to a given cone or cylinder or to a given sphere.

After prefixing therefore the theorems and directions (ἐπιτάγματα) which are necessary for the proof of them, I will then proceed to expound the propositions themselves to you. Farewell.

DEFINITIONS.

If a cone be cut by a plane meeting all the sides [generators] of the cone, the section will be either a circle or a section of an acute-angled cone [an ellipse]. If then the section be a circle, it is clear that the segment cut off from the cone towards the same parts as the vertex of the cone will be a cone. But, if the section be a section of an acute-angled cone [an ellipse], let the figure cut off from the cone towards the same parts as the vertex of the cone be called a **segment of a cone.** Let the **base** of the segment be defined as the plane comprehended by the section of the acute-angled cone, its **vertex** as the point which is also the vertex of the cone, and its **axis** as the straight line joining the vertex of the cone to the centre of the section of the acute-angled cone.

And if a cylinder be cut by two parallel planes meeting all the sides [generators] of the cylinder, the sections will be either circles or sections of acute-angled cones [ellipses] equal and similar to one another. If then the sections be circles, it is clear that the figure cut off from the cylinder between the parallel planes will be a cylinder. But, if the sections be sections of acute-angled cones [ellipses], let the figure cut off from the cylinder between the parallel planes be called a **frustum** (τόμος) **of a cylinder.** And let the **bases** of the frustum be defined as the planes comprehended by the sections of the acute-angled cones [ellipses], and the **axis** as the straight line joining the centres of the sections of the acute-angled cones, so that the axis will be in the same straight line with the axis of the cylinder."

### Lemma.

*If in an ascending arithmetical progression consisting of the magnitudes $A_1, A_2, \dots A_n$ the common difference be equal to the least term $A_1$, then*

$$n \,.\, A_n < 2(A_1 + A_2 + \dots + A_n),$$

*and* $$> 2(A_1 + A_2 + \dots + A_{n-1}).$$

[The proof of this is given incidentally in the treatise *On Spirals*, Prop. 11. By placing lines side by side to represent the terms of the progression and then producing each so as to make it equal to the greatest term, Archimedes gives the equivalent of the following proof.

If $$S_n = A_1 + A_2 + \dots + A_{n-1} + A_n,$$
we have also $$S_n = A_n + A_{n-1} + A_{n-2} + \dots + A_1.$$
And $$A_1 + A_{n-1} = A_2 + A_{n-2} = \dots = A_n.$$
Therefore $$2S_n = (n+1) A_n,$$
whence $$n \,.\, A_n < 2S_n,$$
and $$n \,.\, A_n > 2S_{n-1}.$$

Thus, if the progression is $a, 2a, \dots na$,

$$S_n = \frac{n(n+1)}{2} a,$$

and $$n^2 a < 2S_n,$$
but $$> 2S_{n-1}.]$$

### Proposition 1.

*If $A_1, B_1, C_1, \dots K_1$ and $A_2, B_2, C_2, \dots K_2$ be two series of magnitudes such that*

$$\left.\begin{array}{l} A_1 : B_1 = A_2 : B_2, \\ B_1 : C_1 = B_2 : C_2, \text{ and so on} \end{array}\right\} \dots\dots\dots\dots (\alpha),$$

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