ON THE EQUILIBRIUM OF PLANES.

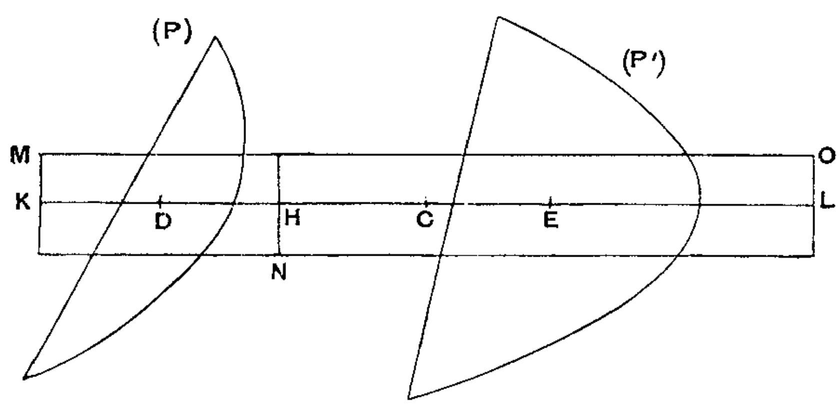
BOOK II.

Proposition 1.

If P, P' be two parabolic segments and D, E their centres of gravity respectively, the centre of gravity of the two segments taken together will be at a point C on DE determined by the relation

$$P : P' = CE : CD^*.$$

In the same straight line with DE measure EH, EL each equal to DC, and DK equal to DH; whence it follows at once that $DK = CE$, and also that $KC = CL$.



* This proposition is really a particular case of Props. 6, 7 of Book I. and is therefore hardly necessary. As, however, Book II. relates exclusively to parabolic segments, Archimedes' object was perhaps to emphasize the fact that the magnitudes in I. 6, 7 might be parabolic segments as well as rectilinear figures. His procedure is to substitute for the segments rectangles of equal area, a substitution which is rendered possible by the results obtained in his separate treatise on the *Quadrature of the Parabola.*

Apply a rectangle MN equal in area to the parabolic segment P to a base equal to KH, and place the rectangle so that KH bisects it, and is parallel to its base.

Then D is the centre of gravity of MN, since $KD = DH$.

Produce the sides of the rectangle which are parallel to KH, and complete the rectangle NO whose base is equal to HL. Then E is the centre of gravity of the rectangle NO.

Now
$$\begin{aligned}(MN) : (NO) &= KH : HL\\ &= DH : EH\\ &= CE : CD\\ &= P : P'.\end{aligned}$$

But
$$(MN) = P.$$

Therefore
$$(NO) = P'.$$

Also, since C is the middle point of KL, C is the centre of gravity of the whole parallelogram made up of the two parallelograms (MN), (NO), which are equal to, and have the same centres of gravity as, P, P' respectively.

Hence C is the centre of gravity of P, P' taken together.

Definition and lemmas preliminary to Proposition 2.

"If in a segment bounded by a straight line and a section of a right-angled cone [a parabola] a triangle be inscribed having the same base as the segment and equal height, if again triangles be inscribed in the remaining segments having the same bases as the segments and equal height, and if in the remaining segments triangles be inscribed in the same manner, let the resulting figure be said to be **inscribed in the recognised manner** (γνωρίμως ἐγγράφεσθαι) in the segment.

And it is plain

(1) that *the lines joining the two angles of the figure so inscribed which are nearest to the vertex of the segment, and the next*

pairs of angles in order, will be parallel to the base of the segment,

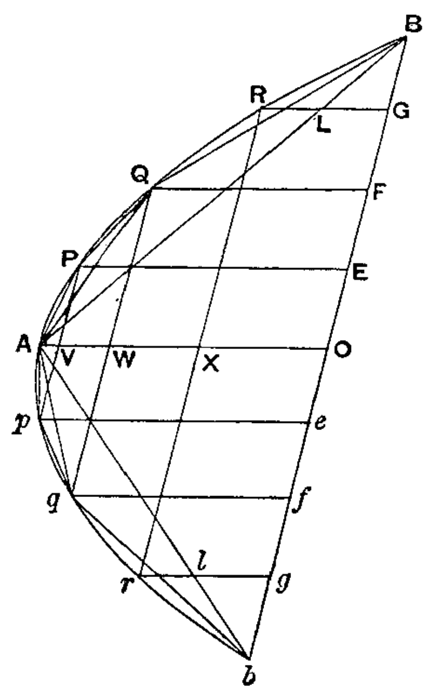
(2) that *the said lines will be bisected by the diameter of the segment,* and

(3) that *they will cut the diameter in the proportions of the successive odd numbers, the number one having reference to [the length adjacent to] the vertex of the segment.*

And these properties will have to be proved in their proper places (ἐν ταῖς τάξεσιν)."

[The last words indicate an intention to give these propositions in their proper connexion with systematic proofs; but the intention does not appear to have been carried out, or at least we know of no lost work of Archimedes in which they could have appeared. The results can however be easily derived from propositions given in the *Quadrature of the Parabola* as follows.

(1) Let $BRQPApqrb$ be a figure inscribed 'in the recognised manner' in the parabolic segment BAb of which Bb is the base, A the vertex and AO the diameter.



Bisect each of the lines BQ, BA, QA, Aq, Ab, qb, and through the middle points draw lines parallel to AO meeting Bb in G, F, E, e, f, g respectively.

These lines will then pass through the vertices R, Q, P, p, q, r of the respective parabolic segments [*Quadrature of the Parabola*, Prop. 18], i.e. through the angular points of the inscribed figure (since the triangles and segments are of equal height).

Also $BG = GF = FE = EO$, and $Oe = ef = fg = gb$. But $BO = Ob$, and therefore all the parts into which Bb is divided are equal.

If now AB, RG meet in L, and Ab, rg in l, we have

$$BG : GL = BO : OA, \text{ by parallels,}$$
$$= bO : OA$$
$$= bg : gl,$$

whence $GL = gl$.

Again [*ibid.*, Prop. 4]

$$GL : LR = BO : OG$$
$$= bO : Og$$
$$= gl : lr;$$

and, since $GL = gl$, $LR = lr$.

Therefore GR, gr are equal as well as parallel.

Hence $GRrg$ is a parallelogram, and Rr is parallel to Bb.

Similarly it may be shown that Pp, Qq are each parallel to Bb.

(2) Since $RGgr$ is a parallelogram, and RG, rg are parallel to AO, while $GO = Og$, it follows that Rr is bisected by AO.

And similarly for Pp, Qq.

(3) Lastly, if V, W, X be the points of bisection of Pp, Qq, Rr,

$$AV : AW : AX : AO = PV^2 : QW^2 : RX^2 : BO^2$$
$$= 1 : 4 : 9 : 16,$$

whence $\qquad AV : VW : WX : XO = 1 : 3 : 5 : 7.]$

Proposition 2.

If a figure be 'inscribed in the recognised manner' in a parabolic segment, the centre of gravity of the figure so inscribed will lie on the diameter of the segment.

For, in the figure of the foregoing lemmas, the centre of gravity of the trapezium $BRrb$ must lie on XO, that of the trapezium $RQqr$ on WX, and so on, while the centre of gravity of the triangle PAp lies on AV.

Hence the centre of gravity of the whole figure lies on AO.

Proposition 3.

If BAB', bab' be two similar parabolic segments whose diameters are AO, ao respectively, and if a figure be inscribed in each segment 'in the recognised manner,' the number of sides in each figure being equal, the centres of gravity of the inscribed figures will divide AO, ao in the same ratio.

[Archimedes enunciates this proposition as true of *similar* segments, but it is equally true of segments which are not similar, as the course of the proof will show.]

Suppose $BRQPAP'Q'R'B'$, $brqpap'q'r'b'$ to be the two figures inscribed 'in the recognised manner.' Join PP', QQ', RR' meeting AO in L, M, N, and pp', qq', rr' meeting ao in l, m, n.

Then [Lemma (3)]

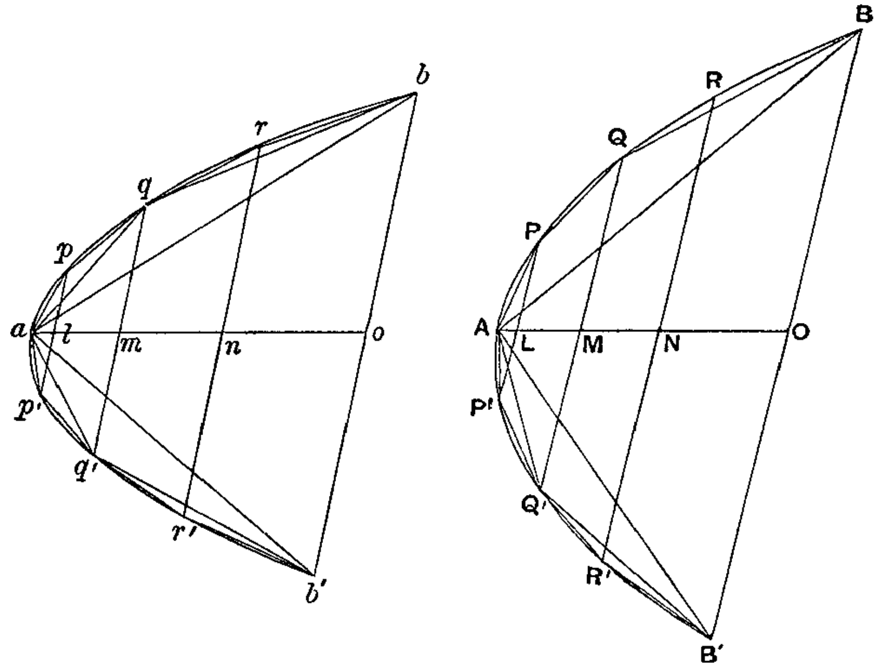
$$AL : LM : MN : NO$$
$$= 1 : 3 : 5 : 7$$
$$= al : lm : mn : no,$$

so that AO, ao are divided in the same proportion.

Also, by reversing the proof of Lemma (3), we see that

$$PP' : pp' = QQ' : qq' = RR' : rr' = BB' : bb'.$$

Since then $RR' : BB' = rr' : bb'$, and these ratios respectively determine the proportion in which NO, no are divided by the centres of gravity of the trapezia $BRR'B'$, $brr'b'$ [I. 15], it follows that the centres of gravity of the trapezia divide NO, no in the same ratio.



Similarly the centres of gravity of the trapezia $RQQ'R'$, $rqq'r'$ divide MN, mn in the same ratio respectively, and so on.

Lastly, the centres of gravity of the triangles PAP', pap' divide AL, al respectively in the same ratio.

Moreover the corresponding trapezia and triangles are, each to each, in the same proportion (since their sides and heights are respectively proportional), while AO, ao are divided in the same proportion.

Therefore the centres of gravity of the complete inscribed figures divide AO, ao in the same proportion.

Proposition 4.

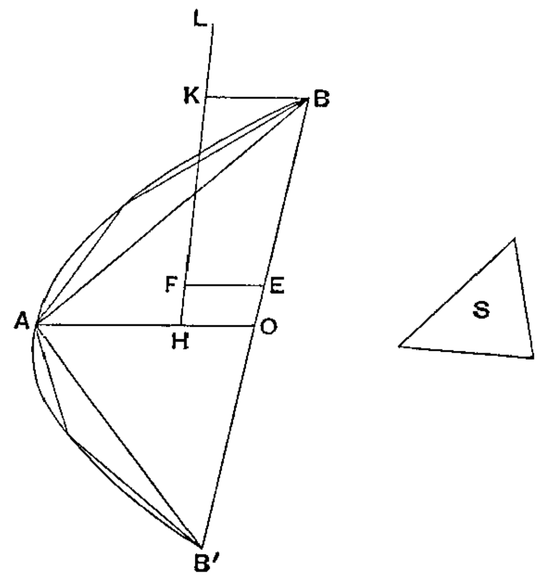
The centre of gravity of any parabolic segment cut off by a straight line lies on the diameter of the segment.

Let BAB' be a parabolic segment, A its vertex and AO its diameter.

Then, if the centre of gravity of the segment does not lie on AO, suppose it to be, if possible, the point F. Draw FE parallel to AO meeting BB' in E.

Inscribe in the segment the triangle ABB' having the same vertex and height as the segment, and take an area S such that

$$\triangle ABB' : S = BE : EO.$$



We can then inscribe in the segment 'in the recognised manner' a figure such that the segments of the parabola left over are together less than S. [For Prop. 20 of the *Quadrature of the Parabola* proves that, if in any segment the triangle with the same base and height be inscribed, the triangle is greater than half the segment; whence it appears that, each time that we increase the number of the sides of the figure inscribed 'in the recognised manner,' we take away more than half of the remaining segments.]

Let the inscribed figure be drawn accordingly; its centre of gravity then lies on AO [Prop. 2]. Let it be the point H.

Join HF and produce it to meet in K the line through B parallel to AO.

Then we have

$$\begin{aligned}(\text{inscribed figure}) : (\text{remainder of segmt.}) &> \triangle ABB' : S \\ &> BE : EO \\ &> KF : FH.\end{aligned}$$

Suppose L taken on HK produced so that the former ratio is equal to the ratio $LF : FH$.

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