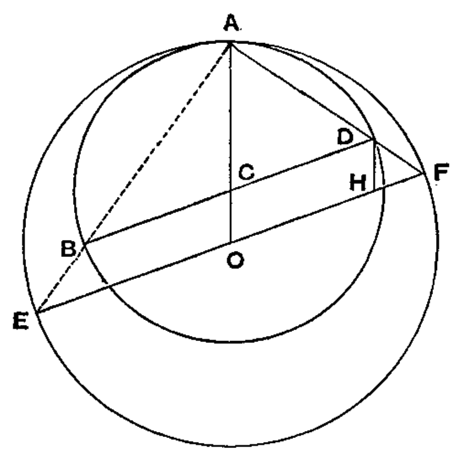
BOOK OF LEMMAS.

Proposition 1.

If two circles touch at A, and if BD, EF be parallel diameters in them, ADF is a straight line.

[The proof in the text only applies to the particular case where the diameters are perpendicular to the radius to the point of contact, but it is easily adapted to the more general case by one small change only.]

Let O, C be the centres of the circles, and let OC be joined and produced to A. Draw DH parallel to AO meeting OF in H.



Then, since $\qquad OH = CD = CA,$

and $\qquad OF = OA,$

we have, by subtraction,

$$HF = CO = DH.$$

Therefore $\qquad \angle HDF = \angle HFD.$

Thus both the triangles CAD, HDF are isosceles, and the third angles ACD, DHF in each are equal. Therefore the equal angles in each are equal to one another, and

$$\angle ADC = \angle DFH.$$

Add to each the angle CDF, and it follows that

$$\angle ADC + \angle CDF = \angle CDF + \angle DFH$$
$$= \text{(two right angles)}.$$

Hence ADF is a straight line.

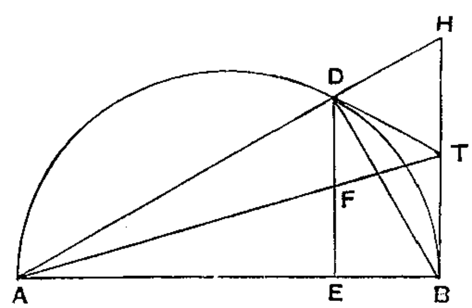
The same proof applies if the circles touch externally*

Proposition 2.

Let AB be the diameter of a semicircle, and let the tangents to it at B and at any other point D on it meet in T. If now DE be drawn perpendicular to AB, and if AT, DE meet in F,

$$DF = FE.$$

Produce AD to meet BT produced in H. Then the angle ADB in the semicircle is right; therefore the angle BDH is also right. And TB, TD are equal.



Therefore T is the centre of the semicircle on BH as diameter, which passes through D.

Hence $$HT = TB.$$

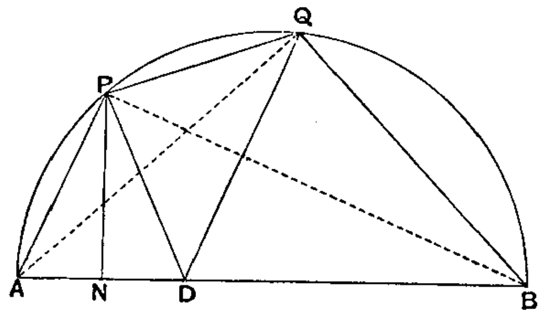
And, since DE, HB are parallel, it follows that $DF = FE$.

* Pappus assumes the result of this proposition in connexion with the ἄρβηλος (p. 214, ed. Hultsch), and he proves it for the case where the circles touch externally (p. 840).

Proposition 3.

Let P be any point on a segment of a circle whose base is AB, and let PN be perpendicular to AB. Take D on AB so that AN = ND. If now PQ be an arc equal to the arc PA, and BQ be joined,

BQ, BD shall be equal.*



Join PA, PQ, PD, DQ.

* The segment in the figure of the MS. appears to have been a semicircle, though the proposition is equally true of any segment. But the case where the segment is a semicircle brings the proposition into close connexion with a proposition in Ptolemy's μεγάλη σύνταξις, I. 9 (p. 31, ed. Halma; cf. the reproduction in Cantor's *Gesch. d. Mathematik*, I. (1894), p. 389). Ptolemy's object is to connect by an equation the lengths of the chord of an arc and the chord of half the arc. Substantially his procedure is as follows. Suppose AP, PQ to be equal arcs, AB the diameter through A; and let AP, PQ, AQ, PB, QB be joined. Measure BD along BA equal to BQ. The perpendicular PN is now drawn, and it is proved that $PA = PD$, and $AN = ND$.

Then $$AN = \tfrac{1}{2}(BA - BD) = \tfrac{1}{2}(BA - BQ) = \tfrac{1}{2}\left(BA - \sqrt{BA^2 - AQ^2}\right).$$

And, by similar triangles, $$AN : AP = AP : AB.$$

Therefore $$AP^2 = AB \,.\, AN$$
$$= \tfrac{1}{2}\left(AB - \sqrt{AB^2 - AQ^2}\right) . AB.$$

This gives AP in terms of AQ and the known diameter AB. If we divide by AB^2 throughout, it is seen at once that the proposition gives a geometrical proof of the formula

$$\sin^2 \frac{\alpha}{2} = \tfrac{1}{2}(1 - \cos \alpha).$$

The case where the segment is a semicircle recalls also the method used by Archimedes at the beginning of the second part of Prop. 3 of the *Measurement of a circle*. It is there proved that, in the figure above,

$$AB + BQ : AQ = BP : PA,$$

or, if we divide the first two terms of the proposition by AB,

$$(1 + \cos \alpha)/\sin \alpha = \cot \frac{\alpha}{2}.$$

Then, since the arcs PA, PQ are equal,

$$PA = PQ.$$

But, since $AN = ND$, and the angles at N are right,

$$PA = PD.$$

Therefore $$PQ = PD,$$

and $$\angle PQD = \angle PDQ.$$

Now, since A, P, Q, B are concyclic,

$$\angle PAD + \angle PQB = \text{(two right angles)},$$

whence $$\angle PDA + \angle PQB = \text{(two right angles)}$$
$$= \angle PDA + \angle PDB.$$

Therefore $$\angle PQB = \angle PDB;$$

and, since the parts, the angles PQD, PDQ, are equal,

$$\angle BQD = \angle BDQ,$$

and $$BQ = BD.$$

Proposition 4.

If AB be the diameter of a semicircle and N any point on AB, and if semicircles be described within the first semicircle and having AN, BN as diameters respectively, the figure included between the circumferences of the three semicircles is "what Archimedes called an ἄρβηλος"; and its area is equal to the circle on PN as diameter, where PN is perpendicular to AB and meets the original semicircle in P.*

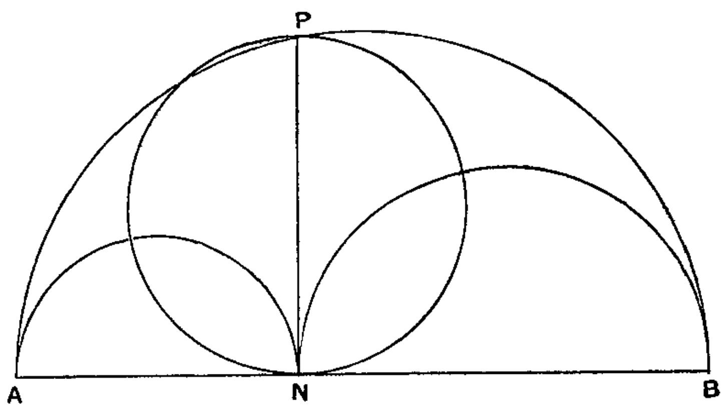
For $$AB^2 = AN^2 + NB^2 + 2AN \cdot NB$$
$$= AN^2 + NB^2 + 2PN^2.$$

But circles (or semicircles) are to one another as the squares of their radii (or diameters).

* ἄρβηλος is literally 'a shoemaker's knife.' Cf. note attached to the remarks on the *Liber Assumptorum* in the Introduction, Chapter II.

Hence

$$(\text{semicircle on } AB) = (\text{sum of semicircles on } AN, NB) + 2\,(\text{semicircle on } PN).$$



That is, the circle on PN as diameter is equal to the difference between the semicircle on AB and the sum of the semicircles on AN, NB, i.e. is equal to the area of the ἄρβηλος.

Proposition 5.

Let AB be the diameter of a semicircle, C any point on AB, and CD perpendicular to it, and let semicircles be described within the first semicircle and having AC, CB as diameters. Then, if two circles be drawn touching CD on different sides and each touching two of the semicircles, the circles so drawn will be equal.

Let one of the circles touch CD at E, the semicircle on AB in F, and the semicircle on AC in G.

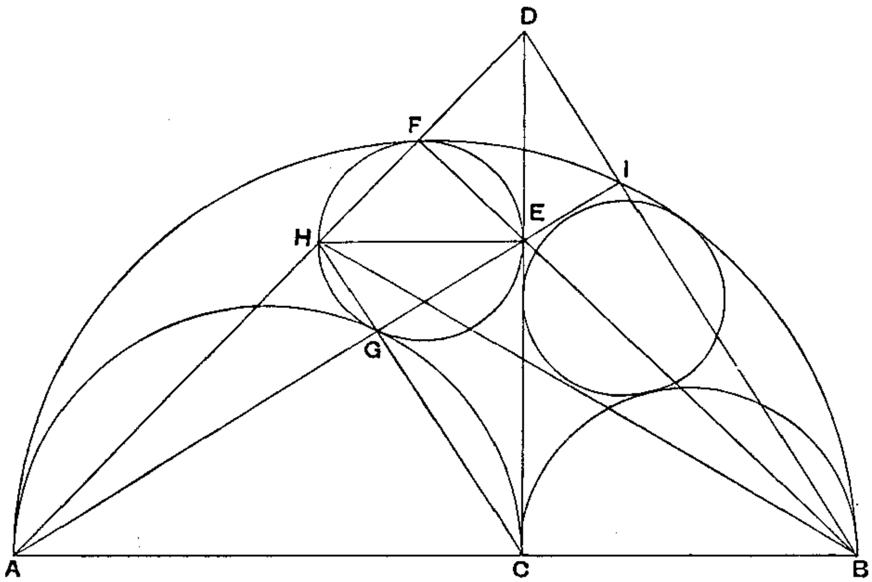
Draw the diameter EH of the circle, which will accordingly be perpendicular to CD and therefore parallel to AB.

Join FH, HA, and FE, EB. Then, by Prop. 1, FHA, FEB are both straight lines, since EH, AB are parallel.

For the same reason AGE, CGH are straight lines.

Let AF produced meet CD in D, and let AE produced meet the outer semicircle in I. Join BI, ID.

Then, since the angles AFB, ACD are right, the straight lines AD, AB are such that the perpendiculars on each from the extremity of the other meet in the point E. Therefore, by the properties of triangles, AE is perpendicular to the line joining B to D.



But AE is perpendicular to BI.

Therefore BID is a straight line.

Now, since the angles at G, I are right, CH is parallel to BD.

Therefore $$AB : BC = AD : DH$$
$$= AC : HE,$$

so that $$AC \,.\, CB = AB \,.\, HE.$$

In like manner, if d is the diameter of the other circle, we can prove that $$AC \,.\, CB = AB \,.\, d.$$

Therefore $d = HE$, and the circles are equal*.

* The property upon which this result depends, viz. that

$$AB : BC = AC : HE,$$

appears as an intermediate step in a proposition of Pappus (p. 230, ed. Hultsch) which proves that, in the figure above,

$$AB : BC = CE^2 : HE^2.$$

The truth of the latter proposition is easily seen. For, since the angle CEH is a right angle, and EG is perpendicular to CH,

$$CE^2 : EH^2 = CG : GH$$
$$= AC : HE.$$

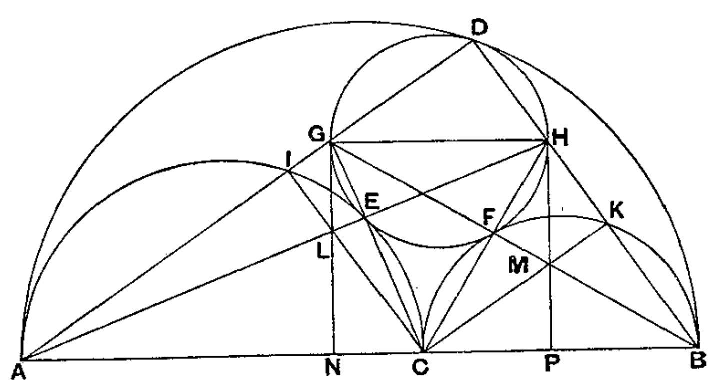
[As pointed out by an Arabian Scholiast Alkauhi, this proposition may be stated more generally. If, instead of one point C on AB, we have two points C, D, and semicircles be described on AC, BD as diameters, and if, instead of the perpendicular to AB through C, we take the radical axis of the two semicircles, then the circles described on different sides of the radical axis and each touching it as well as two of the semicircles are equal. The proof is similar and presents no difficulty.]

Proposition 6.

Let AB, the diameter of a semicircle, be divided at C so that $AC = \frac{3}{2} CB$ [or in any ratio]. Describe semicircles within the first semicircle and on AC, CB as diameters, and suppose a circle drawn touching all three semicircles. If GH be the diameter of this circle, to find the relation between GH and AB.

Let GH be that diameter of the circle which is parallel to AB, and let the circle touch the semicircles on AB, AC, CB in D, E, F respectively.

Join AG, GD and BH, HD. Then, by Prop. 1, AGD, BHD are straight lines.



For a like reason AEH, BFG are straight lines, as also are CEG, CFH.

Let AD meet the semicircle on AC in I, and let BD meet the semicircle on CB in K. Join CI, CK meeting AE, BF

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