THE
METHOD OF ARCHIMEDES

INTRODUCTORY NOTE

From the point of view of the student of Greek mathematics there has been, in recent years, no event comparable in interest with the discovery by Heiberg in 1906 of a Greek MS. containing, among other works of Archimedes, substantially the whole of a treatise which was formerly thought to be irretrievably lost.

The full description of the MS. as given in the preface to Vol. I. (1910) of the new edition of Heiberg's text of Archimedes now in course of publication is—

Codex rescriptus Metochii Constantinopolitani S. Sepulchri monasterii Hierosolymitani 355, 4to.

Heiberg has told the story of his discovery of this MS. and given a full description of it*. His attention having been called to a notice in Vol. IV. (1899) of the Ἱεροσολυμιτικὴ βιβλιοθήκη of Papadopulos Kerameus relating to a palimpsest of mathematical content, he at once inferred from a few specimen lines which were quoted that the MS. must contain something by Archimedes. As the result of inspection, at Constantinople, of the MS. itself, and by means of a photograph taken of it, he was able to see what it contained and to decipher much of the contents. This was in the year 1906, and he inspected the MS. once more in 1908. With the exception of the last leaves, 178 to 185, which are of paper of the 16th century, the MS. is of parchment and contains writings of Archimedes copied in a good hand of the 10th century, in two columns. An attempt was made (fortunately with only partial success) to wash out the old writing, and then the parchment was used again, for the purpose of writing a Euchologion thereon, in the 12th—13th or 13th—14th centuries. The earlier writing appears with more or less clearness on most of the 177 leaves; only 29 leaves are destitute of any trace of such writing; from 9 more it was hopelessly washed off; on a few more leaves only a few words can be made out; and again some 14 leaves have old writing

* *Hermes* XLII. 1907, pp. 235 sq.

upon them in a different hand and with no division into columns. All the rest is tolerably legible with the aid of a magnifying glass. Of the treatises of Archimedes which are found in other MSS., the new MS. contains, in great part, the books *On the Sphere and Cylinder*, almost the whole of the work *On Spirals*, and some parts of the *Measurement of a Circle* and of the books *On the Equilibrium of Planes*. But the important fact is that it contains (1) a considerable proportion of the work *On Floating Bodies* which was formerly supposed to be lost so far as the Greek text is concerned and only to have survived in the translation by Wilhelm von Mörbeke, and (2), most precious of all, the greater part of the book called, according to its own heading, Ἔφοδος and elsewhere, alternatively, Ἐφόδιον or Ἐφοδικόν, meaning *Method*. The portion of this latter work contained in the MS. has already been published by Heiberg (1) in Greek* and (2) in a German translation with commentary by Zeuthen†. The treatise was formerly only known by an allusion to it in Suidas, who says that Theodosius wrote a commentary upon it; but the *Metrica* of Heron, newly discovered by R. Schöne and published in 1903, quotes three propositions from it‡, including the two main propositions enunciated by Archimedes at the beginning as theorems novel in character which the method furnished a means of investigating. Lastly the MS. contains two short propositions, in addition to the preface, of a work called *Stomachion* (as it might be "Neck-Spiel" or "Quäl-Geist") which treated of a sort of Chinese puzzle known afterwards by the name of "loculus Archimedius"; it thus turns out that this puzzle, which Heiberg was formerly disinclined to attribute to Archimedes§, is really genuine.

The *Method*, so happily recovered, is of the greatest interest for the following reason. Nothing is more characteristic of the classical works of the great geometers of Greece, or more tantalising, than the absence of any indication of the steps by which they worked their way to the discovery of their great theorems. As they have come down to us, these theorems are finished masterpieces which leave no traces of any rough-hewn stage, no hint of the method by which they were evolved. We cannot but suppose that the

* *Hermes* XLII. 1907, pp. 243—297.

† *Bibliotheca Mathematica* VII_3, 1906-7, pp. 321—363.

‡ *Heronis Alexandrini opera*, Vol. III. 1903, pp. 80, 17; 130, 15; 130, 25.

§ Vide *The Works of Archimedes*, p. xxii.

Greeks had some method or methods of analysis hardly less powerful than those of modern analysis; yet, in general, they seem to have taken pains to clear away all traces of the machinery used and all the litter, so to speak, resulting from tentative efforts, before they permitted themselves to publish, in sequence carefully thought out, and with definitive and rigorously scientific proofs, the results obtained. A partial exception is now furnished by the *Method*; for here we have a sort of lifting of the veil, a glimpse of the interior of Archimedes' workshop as it were. He tells us how he discovered certain theorems in quadrature and cubature, and he is at the same time careful to insist on the difference between (1) the means which may be sufficient to suggest the truth of theorems, although not furnishing scientific proofs of them, and (2) the rigorous demonstrations of them by irrefragable geometrical methods which must follow before they can be finally accepted as established; to use Archimedes' own terms, the former enable theorems to be *investigated* (θεωρεῖν) but not to be *proved* (ἀποδεικνύναι). The mechanical method, then, used in our treatise and shown to be so useful for the discovery of theorems is distinctly said to be incapable of furnishing proofs of them; and Archimedes promises to add, as regards the two main theorems enunciated at the beginning, the necessary supplement in the shape of the formal geometrical proof. One of the two geometrical proofs is lost, but fragments of the other are contained in the MS. which are sufficient to show that the method was the orthodox method of exhaustion in the form in which Archimedes applies it elsewhere, and to enable the proof to be reconstructed.

The rest of this note will be best understood after the treatise itself has been read; but the essential features of the mechanical method employed by Archimedes are these. Suppose X to be a plane or solid figure, the area or content of which has to be found. The method is to weigh infinitesimal elements of X (with or without the addition of the corresponding elements of another figure C) against the corresponding elements of a figure B, B and C being such figures that their areas or volumes, and the position of the centre of gravity of B, are known beforehand. For this purpose the figures are first placed in such a position that they have, as common diameter or axis, one and the same straight line; if then the infinitesimal elements are sections of the figures made by parallel planes perpendicular (in general) to the axis and cutting the figures, the centres of gravity of all the elements lie at one point or other on the common diameter or axis. This diameter or axis is produced and is imagined to be the bar or lever of a balance. It is sufficient to take the simple case where the elements of X alone are weighed against the elements of another figure B. The elements which correspond to one another are the sections of X and B respectively by any one plane perpendicular (in general) to the diameter or axis and cutting both figures; the elements are spoken of as straight lines in the case of plane figures and as plane areas in the case of solid figures. Although Archimedes calls the elements straight lines and plane areas respectively, they are of course, in the first case, indefinitely narrow strips (areas) and, in the second case, indefinitely thin plane laminae (solids); but the breadth or thickness (dx, as we might call it) does not enter into the calculation because it is regarded as the same in each of the two corresponding elements which are separately weighed against each other, and therefore divides out. The number of the elements in each figure is infinite, but Archimedes has no need to say this; he merely says that X and B are *made up* of *all* the elements in them respectively, i.e. of the straight lines in the case of areas and of the plane areas in the case of solids.

The object of Archimedes is so to arrange the balancing of the elements that the elements of X are all applied at *one point* of the lever, while the elements of B operate at different points, namely where they actually are in the first instance. He contrives therefore to move the elements of X away from their first position and to concentrate them at one point on the lever, while the elements of B are left where they are, and so operate at their respective centres of gravity. Since the centre of gravity of B as a whole is known, as well as its area or volume, it may then be supposed to act as one mass applied at its centre of gravity; and consequently, taking the whole bodies X and B as ultimately placed respectively, we know the distances of the two centres of gravity from the fulcrum or point of suspension of the lever, and also the area or volume of B. Hence the area or volume of X is found. The method may be applied, conversely, to the problem of finding the centre of gravity of X when its area or volume is known beforehand; in this case it is necessary that the elements of X, and therefore X itself, should be weighed *in the places where they are*, and that the figures the elements of which are moved to one single point of the lever, to be weighed there, should be other figures and not X.

The method will be seen to be, not *integration*, as certain geometrical proofs in the great treatises actually are, but a clever device for *avoiding* the particular integration which would naturally be used to find directly the area or volume required, and making the solution depend, instead, upon *another* integration the result of which is already known. Archimedes deals with *moments* about the point of suspension of the lever, i.e. the products of the elements of area or volume into the distances between the point of suspension of the lever and the centres of gravity of the elements respectively; and, as we said above, while these distances are different for all the elements of B, he contrives, by moving the elements of X, to make them the same for all the elements of X in their final position. He assumes, as known, the fact that the sum of the moments of each particle of the figure B acting at the point where it is placed is equal to the moment of the whole figure applied as one mass at one point, its centre of gravity.

Suppose now that the element of X is $u \,.\, dx$, u being the length or area of a section of X by one of a whole series of parallel planes cutting the lever at right angles, x being measured along the lever (which is the common axis of the two figures) from the point of suspension of the lever as origin. This element is then supposed to be placed on the lever at a constant distance, say a, from the origin and on the opposite side of it from B. If $u' \,.\, dx$ is the corresponding element of B cut off by the same plane and x its distance from the origin, Archimedes' argument establishes the equation

$$a \int_h^k u\,dx = \int_h^k x u' dx.$$

Now the second integral is known because the area or volume of the figure B (say a triangle, a pyramid, a prism, a sphere, a cone, or a cylinder) is known, and it can be supposed to be applied as one mass at its centre of gravity, which is also known; the integral is equal to bU, where b is the distance of the centre of gravity from the point of suspension of the lever, and U is the area or content of B. Hence

$$\text{the area or volume of } X = \frac{bU}{a}.$$

In the case where the elements of X are weighed along with the corresponding elements of another figure C against corresponding

END OF SAMPLE TEXT